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TENSOR PRODUCTS OF IRREDUCIBLE FINITE-DIMENSIONAL LOCALLY BOUNDED PSEUDOREPRESENTATIONS OF CONNECTED SIMPLE LIE GROUPS

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Abstract. It is proved that the tensor product of any two irreducible finite-dimensional locally bounded pseudorepresentations of a connected simple Lie group is a quasirepresentation if and only if the group has finite center, i.e., either the group is not Hermitian symmetric or it is Hermitian symmetric and not simply connected.

§ 1. Introduction

For the definitions, notation, and generalities concerning pseudocharacters, quasicharacters, pseudorepresentations, and quasirepresentations, and also for the definition of the Guichardet–Wigner pseudocharacter on a connected simply connected Hermitian symmetric simple Lie group, see [1]–[3].

Obviously, the tensor product of two reducible pseudorepresentations of a connected simply connected Hermitian symmetric simple Lie group need not be a quasirepresentation. Indeed, let G be such a group and let f be a Guichardet–Wigner pseudocharacter on G. Then the mapping $\pi\colon g \mapsto \left(\begin{smallmatrix} 1 & f(g) \\ 0 & 1 \end{smallmatrix}\right)$, $g \in G$, is a reducible pseudorepresentation of G. One of the matrix entries of the tensor square of π is $f^2(g)$, which is not a quasicharacter, since $f(g^2) - 2f(g)$ is bounded, $f(g^2) = 2f(g) + c(g)$, where $c(g)$ is bounded, and

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hence $f^2(g^2) - 2f^2(g) = 2f^2(g) + 4f(g)c(g) + c(g)^2$, where the right-hand side is quadratic with respect to $f(g)$ and hence unbounded together with f.

Thus, the tensor square of π is not a quasirepresentation.

This suggests the idea to study the tensor products of *irreducible* pseudorepresentations of the groups under consideration.

§ 2. Preliminaries

As was proved in [4], if G is a connected simple Lie group and π be an irreducible locally bounded finite-dimensional finally precontinuous pseudorepresentation of G, the the following dichotomy holds.

If the center of G is finite, then π is an ordinary irreducible continuous finite-dimensional representation of G.

If the center of G is infinite, then π is defined either by an irreducible ordinary continuous finite-dimensional representation of G or by some one-dimensional Guichardet–Wigner pseudorepresentation (i.e., a mapping of the form $g \to \exp(irf(g))$, $g \in L$, for some $r \in \mathbb{R}$, where f is a Guichardet–Wigner pseudocharacter on G).

§ 3. Main theorem

Theorem 1. *The tensor product of any two irreducible finite-dimensional locally bounded pseudorepresentations of a connected simple Lie group is a quasirepresentation if and only if the group has finite center, i.e., either the group is not Hermitian symmetric or it is Hermitian symmetric and not simply connected.*

Proof. It suffices to prove that, if G is a connected simply connected Hermitian symmetric Lie group, then the tensor product of a nontrivial Guichardet–Wigner pseudorepresentation and a nontrivial (unbounded) ordinary continuous finite-dimensional representation of G is not a quasirepresentation.

Let ρ be an unbounded ordinary irreducible continuous finite-dimensional representation of G and let $\theta\colon g \to \exp(irf(g))$, $g \in G$, for some $r \in \mathbb{R}$, where f is a Guichardet–Wigner pseudocharacter on G, be a nontrivial one-dimensional Guichardet–Wigner pseudorepresentation of G. Multiplying f by a positive constant, we may assume that $|f(g) + f(h) - f(gh)| \leq \pi/2$ for all $g, h \in G$. The tensor product of ρ and θ is the ordinary product of a numerical function and an operator-valued function ρ. Consider the difference

$$
\begin{aligned}
\Delta(g,h) &= \theta(g)\rho(g)\theta(h)\rho(h) - \theta(gh)\rho(gh) \\
&= \big(\theta(g)\theta(h) - \theta(gh)\big)\rho(gh) = \delta(g,h)\rho(gh), \qquad (1)
\end{aligned}
$$

where

$$
\begin{aligned}
\delta(g,h) &= \theta(g)\theta(h) - \theta(gh) \\
&= \exp(irf(g))\exp(irf(h)) - \exp(irf(gh)),
\end{aligned}
$$

and hence

$$
\begin{aligned}
|\delta(g,h)| &= |\exp(irf(g))\exp(irf(h)) - \exp(irf(gh))| \\
&= |\exp(ir(f(g)+f(h)-f(gh))) - 1|. \qquad (2)
\end{aligned}
$$

Let $x \in G$ be such that the norm of $\rho(x^n)$ is infinitely large as $n \to \infty$ (there are many elements x with this property). Consider the difference $f(x^n yz) - f(x^n y)$ for $y, z \in G$. If

$$
\lim_{n\to\infty} (f(x^n yz) - f(x^n y)) = f(z)
$$

for all $y, z \in G$, then

$$
\begin{aligned}
f(yz) = \lim_{n\to\infty} (f(x^n yz) - f(x^n)) &= \lim_{n\to\infty} (f(x^n yz) - f(x^n y)) \\
&+ \lim_{n\to\infty} (f(x^n y) - f(x^n)) = f(z) + f(y),
\end{aligned}
$$

which is wrong. Therefore, the sequence $f(x^n yz) - f(x^n y) - f(z)$ does not tend to zero for some $y, z \in G$, and there is a subsequence $f(x^{n_k} yz) - f(x^{n_k} y) - f(z)$ whose absolute value is greater than a positive constant. Combine (1) and (2) for $g = x^{n_k} y$ and $h = z$. This gives $\|\Delta(x^{n_k} y, z)\| = |\exp(ir(f(x^{n_k} y) + f(z) - f(x^{n_k} yz))) - 1|\|\rho(x^{n_k})\rho(yz)\|$. It follows that the product on the right-hand side can be arbitrarily large, and thus the product $\theta\rho$ is not a quasirepresentation.

§ 4. Concluding remarks

Thus, for the case in which a group G in question is a connected simply connected Hermitian symmetric simple Lie group, the families of ordinary continuous finite-dimensional irreducible representations of G and of the Guichardet–Wigner pseudorepresentations of G have in common only the trivial representation and form two separate families for which the tensor products of these families are pseudorepresentations. The corresponding multiplication table of tensor products turns out to be only partial.

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